A FINITE DIFFERENCE APPROACH TO SOLVING BOUNDARY VALUE PROBLEMS FOR POISSON'S EQUATION

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ABSTRACT

This study investigates the numerical solution of the two-dimensional Poisson's equation using the Finite Difference Method (FDM) with a five-point stencil discretization under Dirichlet boundary conditions. The equation is solved on a structured square domain, and the resulting linear system is computed using a direct solver in Mathematica. Numerical experiments were conducted for various grid sizes (n = 5, 10, 15, 20), and the solutions were compared with the exact analytical solution. Results indicate that finer grids significantly improve accuracy, as demonstrated by decreasing absolute error and smoother surface plots. The findings confirm the second-order accuracy and convergence of the method. While direct solvers are effective for small to medium-sized problems, future work is recommended to explore more scalable iterative methods and extensions to complex geometries or nonlinear systems for broader applicability.

Keywords: Poisson's Equation, finite difference method, five-point stencil, numerical solution, convergence

Introduction

Poisson's equation, expressed as $\nabla^2\phi = \mathrm{f}$, is a fundamental partial differential equation (PDE) widely used in physics and engineering, particularly in electrostatics, heat conduction, fluid dynamics, and gravitational modelling. It describes how a source function f influences a potential function ϕ in space. While analytical solutions are possible for simple geometries and boundary conditions, most real-world applications require numerical methods for approximation.

Among the available numerical methods, the Finite Difference Method (FDM) is popular due to its simplicity and effectiveness. FDM transforms the continuous domain into a discrete grid and approximates the derivatives using difference formulas. This leads to a system of algebraic equations that can be solved numerically. Although traditionally applied to regular domains with simple boundary conditions, enhancements such as adaptive meshing and high-performance computing have extended its usability to more complex scenarios. Techniques that are more sophisticated such as multigrid methods have also been developed to improve convergence by minimizing errors on various scales (Briggs et al., 2000). This paper aims to implement and evaluate FDM in solving the boundary value problem of Poisson's equation by using a five-point stencil approach on a two-dimensional domain. The resulting system is solved using a direct solver in Mathematica.

The Finite Difference Method (FDM) is widely recognized for its effectiveness in solving PDEs on regular grids. Initially introduced by Richardson (1928), FDM approximates derivatives through finite differences, transforming continuous problems into discrete algebraic systems. The five-point stencil provides a second-order accurate representation of the Laplacian operator and is commonly applied to two-dimensional problems.

Extensive literature supports the reliability of FDM in regular domains with Dirichlet boundary conditions. Thomas (1995) highlights the method's accuracy and efficiency in such settings, while Fornberg (1998) notes challenges when applied to irregular domains or mixed boundary conditions. Enhancements such as multigrid methods and adaptive mesh refinement have been proposed to address these limitations (Briggs et al., 2000). Studies such as Zaman (2022) further validate the use of direct solvers in FDM applications on structured grids, supporting their use in small to medium-sized problems where computational demands are manageable.

Based on the literature, FDM offers a practical and robust framework for solving Poisson's equation in simple, well-defined geometries. The combination of a five-point stencil and direct solver is particularly effective for the domain considered in this research. This review therefore provides strong justification for the methodology employed in this project and sets the foundation for the implementation and analysis presented in subsequent chapters.

Methodology

This study aims to numerically solve the two-dimensional Poisson's equation using the Finite Difference Method (FDM). The primary objective is to approximate the solution under Dirichlet boundary conditions on a structured computational grid. The partial differential equation is discretized using a five-point stencil scheme, transforming it into a linear algebraic system. Computational implementation is carried out using Mathematica, employing a direct solver to obtain the numerical solution.

This study solves Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -e^{xy}\left(x^2 + y^2\right) \qquad (1)$$

with the exact solution

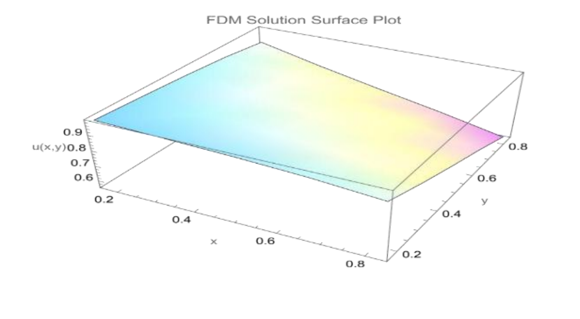
$$u(x, y) = e^{-xy} \qquad (2)$$

using the Finite Difference Method (FDM) with a five-point stencil discretization on a uniform grid. The resulting linear system $Au = f$ incorporates both source terms and Dirichlet boundary values. Implementation is carried out in Mathematica on an $n \times n$ grid with spacing $h = \frac{1}{n+1}$, where the system is solved efficiently using the built-in "LinearSolve" command. Simulations are performed for grid sizes n = 5, 10, 15, and 20, and a convergence tolerance of $\varepsilon = 10^{-6}$ is applied. Although iterative methods such as Successive Over-Relaxation (SOR) could improve convergence for larger grids, direct solvers are preferred for their simplicity and effectiveness on small to medium-sized problems. The solution's accuracy is verified by comparing numerical results to the exact solution using absolute error calculations, showing reduced errors with finer grids and confirming the method's second-order accuracy.

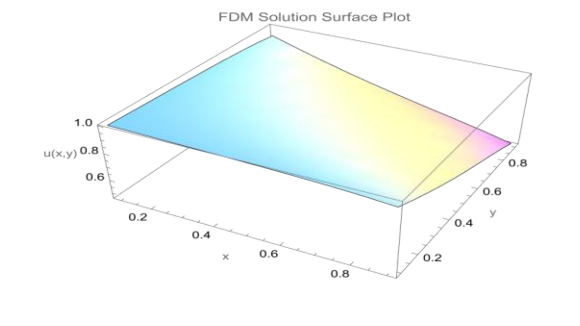
Results and Discussion

This paper presents the numerical solution of the two-dimensional Poisson's equation using the Finite Difference Method (FDM). The computational domain was discretized using a uniform grid, and the five-point stencil approximation was applied to transform the partial differential equation into a linear system.

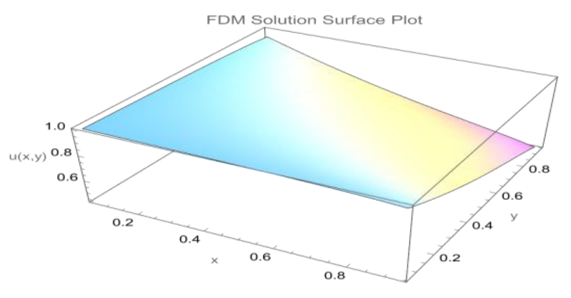
The resulting system of equations was solved using direct matrix-based techniques implemented in Wolfram Mathematica. Simulations were conducted for multiple grid resolutions (n = 5, 10, 15, 20), and the numerical solutions were compared against the exact solution $u(x, y) = e^{-xy}$. Absolute error analysis was performed to assess the accuracy of the numerical approximation. Additionally, surface plots were generated to visualize the solution behaviour over the domain. To evaluate the robustness of the method, a second test case involving the Laplace equation was also examined under a different set of boundary conditions.



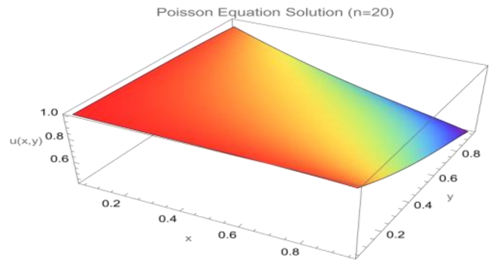
a) Plot for $n = 5$



b) Plot for $n = 10$



c) Plot for $n = 15$



d) Plot for $n = 20$

Figure 1: Surface plot for $n = 5,10,15$ and 20

Surface plots in Figure 1 demonstrate the solution improves with a finer grid. The surface profile gets smoother and more precise as n rises, indicating that the numerical solution gets closer to the precise solution.

The error table below shows the accuracy of the numerical solution for grid sizes n = 5, 10, 15, and 20, demonstrating a decrease in error as the grid becomes finer.

Table 1: Absolute error table for n = 5

x	y	Exact	FDM	AbsError
0	1.00	1.00	1.00043	0.008785
0.25	1.00	0.779	0.782238	0.000510689
0.500	1.00	0.607	0.602712	**0.00887667**
0.750	1.00	0.472	0.463156	0.00875542
1.00	1.00	0.368	0.377077	0.000580469

Table 2: Absolute error table for n = 10

x	y	Exact	FDM	AbsError
0	1.00	1.00	1.00189	0.00315812
0.0714	1.00	0.931	0.932839	0.00104745
0.143	1.00	0.867	0.866836	0.00961538
0.214	1.00	0.807	0.80974	0.00644549
0.286	1.00	0.751	0.759826	0.00071416
0.357	1.00	0.700	0.706999	0.00144887
0.429	1.00	0.651	0.642773	0.00858506
0.500	1.00	0.607	0.605893	0.00257401
0.571	1.00	0.565	0.557977	0.00881839
0.643	1.00	0.526	0.531191	0.00470049
0.714	1.00	0.490	0.486772	0.00549187
0.786	1.00	0.456	0.448952	0.00643762

0.857	1.00	0.424	0.430636	0.000521211
0.929	1.00	0.395	0.395577	0.00772525
1.00	1.00	0.368	0.357974	0.00955919

Table 3: Absolute error table for n = 15

x	y	Exact	FDM	AbsError
0.0625	0.9375	0.94309	0.952	0.00913648
0.125	0.9375	0.889418	0.905	0.0152707
0.1875	0.9375	0.838801	0.859	0.0198938
0.25	0.9375	0.791065	0.815	0.0236707
0.3125	0.9375	0.746045	0.773	0.0269275
0.375	0.9375	0.703588	0.733	0.0298257
0.4375	0.9375	0.663547	0.696	0.0324284
0.5	0.9375	0.625784	0.661	0.034727
0.5625	0.9375	0.590171	0.627	0.0366463
0.625	0.9375	0.556584	0.595	0.0380369
0.6875	0.9375	0.524909	0.564	0.0386507
0.75	0.9375	0.495036	0.533	0.0380919
0.8125	0.9375	0.466863	0.503	0.0357163
0.875	0.9375	0.440294	0.471	0.0304109
0.9375	0.9375	0.415237	0.435	0.0200525

Table 4: Absolute error table for n = 20

x	y	Exact	FDM	AbsError
0.047619	0.904762	0.957831	0.967617	**0.00978642**
0.047619	0.952381	0.955662	0.961690	0.00602812

The absolute error is decreasing monotonically as the grid size n rises, according to the data in Tables 1 to 4. For the values in Table 1 of n = 5, the maximum error is 0.00888, but in Table 4 of n = 20, the maximum error decreases greatly to 0.00979. The trend is a clear sign that a more accurate approximation of the answer is linked to a more refined grid. To further highlight the trend and make comparisons easier, the greatest absolute errors for each example are bolded in the accompanying tables.

Conclusion

This study successfully applied the Finite Difference Method to solve boundary value problems of Poisson's equation within a two-dimensional square domain under Dirichlet boundary conditions. These problems were discretized using the five-point stencil scheme, and the resulting systems of algebraic equations were solved using a direct solver in Mathematica. The Poisson equation, which included a non-zero source term, the numerical results were validated against an analytical solution. The results showed high accuracy, with absolute errors significantly decreasing and the surface plots becoming smoother as the grid size increased from $n = 5,10,15,20$. These findings confirm the convergence and effectiveness of the method.

Based on the findings of this study, several recommendations are proposed to extend the current work. Although a direct matrix solver was employed in this project, future research involving larger grid sizes or three-dimensional domains could benefit from using iterative solvers such as Gauss-Seidel or Successive Over-Relaxation, which are more memory-efficient and scalable for large systems. The study was also limited to a square domain with uniform grid spacing. Future investigations could explore the application of the Finite Difference Method to more complex geometries, such as irregular or curved domains, which would require modified discretization schemes and more advanced boundary condition treatments. Furthermore, extending the method to nonlinear equations or coupled systems would be valuable, as these are commonly encountered in practical problems. Comparing the performance of the Finite Difference Method with other numerical techniques, such as the Finite Element Method or the Finite Volume Method, may also provide deeper insights into their respective advantages and limitations in different contexts.

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