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SOLVING HEAT EQUATION USING FINITE VOLUME METHOD AND CRANK-NICOLSON METHOD

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ABSTRACT

The heat equation which is a significant partial differential equation depicting temperature distribution of a given domain at a given time is used in several engineering and scientific problems. This research rises in the context of the heat equation solving, based on the Finite Volume Method and the Crank-Nicolson Method with the purpose to enhance the numerical stability, accuracy, and computational cost reduction. The usage of maple tools to calculate temperature distributions through time with different step sizes is demonstrated in this research. The main goals include evaluating the performance of these methods in terms of accuracy and convergence, and assessing their capabilities in dealing with nonlinearity and heterogeneity of the material. The study then uses these numerical techniques to get its solutions that are then compared with those of analytical solutions. Some new insight called into question indicate that the Crank-Nicolson method is more accurate and stable when in relation to complicated shapes while the Finite Volume Method is more effective in terms of conservative quantity like heat over finite volumes. Such outcomes suggest the applicability of these methods in the development of heat transfer research, stating the ways of utilizing it in practical engineering problems regarding the disposal of computational time and increasing the precision of calculations.

Keywords: Finite Volume Method, Crank-Nicolson, Heat Equation

Introduction

The heat equation, a partial differential equation describing temperature distribution in a heat-conducting body, is crucial in many fields such as physics, engineering, and materials science. Analytical solutions are often impractical for complex geometries and boundary conditions, necessitating numerical methods like the Finite Difference Method (FDM) and Finite Volume Method (FVM). This research aims to solve the heat equation using the Crank-Nicolson method within FDM and the FVM, comparing their accuracy and stability. FDM, particularly the Crank-Nicolson approach, is known for its stability and second-order accuracy, making it suitable for complex geometries (Mojumder, 2023). FVM, adept at handling irregular shapes, integrates over control volumes and is precise with minimal error (Saptaningtyas, 2018). The study focuses on identifying efficient methods that balance computational resources and accuracy, especially for non-linear and heterogeneous material properties (Xiang, 2018). Using numerical computing environments like Maple, this research evaluates these methods under various step sizes and boundary conditions, aiming to determine the most suitable approach for solving heat equations in practical scenarios. However, limitations include challenges in achieving stability and accurate results with the Crank-Nicolson method and the impact of boundary and initial condition assumptions (Zhu, 2021). Further research may explore using different boundary conditions to enhance the reliability and applicability of these numerical techniques in real-life situations.

Methodology

Finite Volume Method using Explicit Method

Heat or thermal energy is a form of energy found in solids, liquids, and gases. Its units can be converted to joules or calories, with 1 cal = 4.184 J. The heat equation governs the transfer of thermal energy in solid and liquid phases, focusing on isolated points (Herbin, 2023).

The metal rod of length *l* is divided into small control volumes, were heat travels from hotter to cooler regions under constant conduction through insulation.

The heat equation is defined as:

$$\frac{\partial}{\partial t}(e(x,t)) = \frac{w}{A\,\Delta x}$$

where e(x,t) is thermal energy density, *w* is heat flow per unit area, *A* is area, and Δx is rod length.

Numerical Solution of One-Dimensional Heat Equation by Crank-Nicolson Method

We solve the one-dimensional heat equation using the Crank-Nicolson method, essential for homogeneous boundary-value problems with Dirichlet conditions (Islam, 2018).
The heat equation is:

$$\frac{\partial u(x,t)}{\partial t} = \beta \frac{\partial^2 u(x,t)}{\partial x^2}$$

with initial condition u(x,0) = f(x) and boundary conditions u(0,t) = T_0, u(l,t)=T_1.
The Crank-Nicolson finite difference method is given by:

$$\delta\left(u_{j+1}^{i+1} + u_{j+1}^{i-1}\right) - \left((2\delta + 2)\cdot u_{j+1}^{i}\right) = (2\delta + 2)\cdot u_j^i - \delta\left(u_j^{i+1} - u_j^{i-1}\right)$$

Where $\delta = \frac{kh^2}{\beta}$

Unsteady State Problem of Explicit Method

Solve the heat equation using an explicit method:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad t \geq 0$$

with initial condition u(*x*,0) = *x* (1 − *x*) and boundary conditions u(0,t) = u(1,t) = 0.

Unsteady State Problem of Implicit Method

Solve the heat equation using an implicit method:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad t \geq 0$$

with initial condition u(x, 0) = x(2 − x) and boundary conditions u(0, t) = u(2, t) = 0.

Results

Finite Volume Method

Although explicit methods require small time steps to maintain numerical stability typically governed by the (Courants-Friedrichs-Lewy (CFL) condition), this constraint can be managed in many practical applications where high temporal resolution is acceptable.

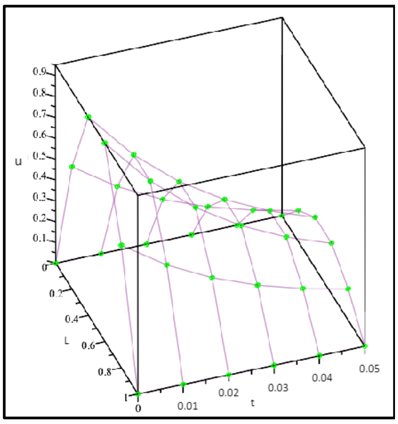

Figure 1: The grid function of Finite Volume Method simulation with spatial step size (h) = 0.2 and number of spatial divisions (NX) = 5.

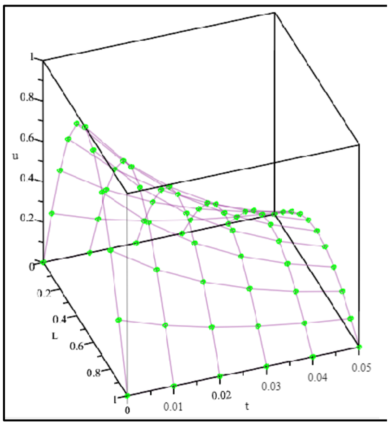

Figure 2: The grid function of Finite Volume Method simulation with spatial step size (h) = 0.1 and number of spatial divisions (NX) = 10.

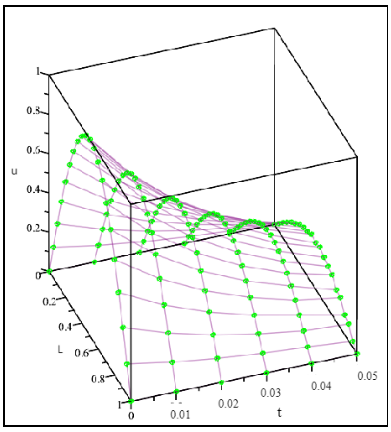

Figure 3: The grid function of Finite Volume Method simulation with spatial step size (h) = 0.05 and number of spatial divisions (NX) = 20.

Explicit Crank-Nicolson Method

The highly known explicit Crank-Nicolson method is preferred widely for its simplicity both in terms of understanding and implementation, calculating the temperatures at the grid points depending on the temperature there in the previous time step, with maximum time of **0.05 seconds.**

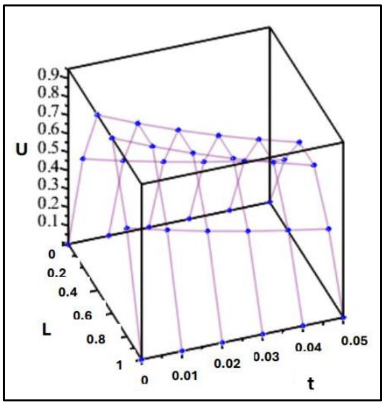

Figure 4: Crank-Nicolson with spatial step size, h = 0.2 and number of spatial divisions, NX = 5

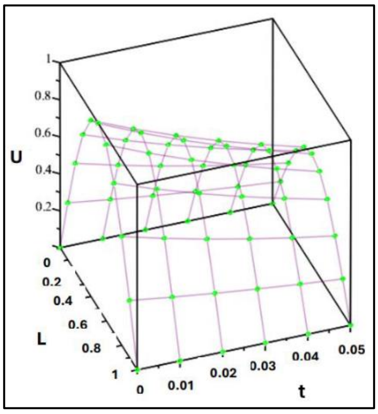

Figure 5: Crank-Nicolson with spatial step size, h = 0.1 and number of spatial divisions, NX = 10

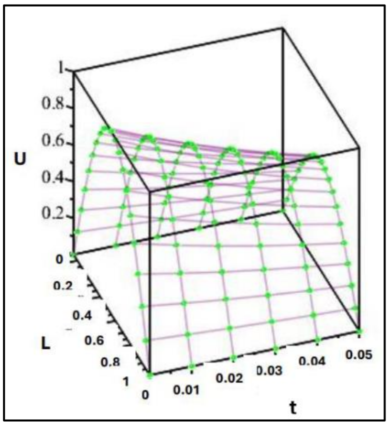

Figure 6: Crank-Nicolson with spatial step size, h = 0.05 and number of spatial divisions, NX = 20

Discussion

The finite volume method is often solved using the explicit method due to its straightforward implementation, involving simple algebraic updates at each time step (Stabile, 2018). Despite requiring small time steps to maintain numerical stability governed by the Courant-Friedrichs-Lewy (CFL) condition, this constraint can be managed in many practical applications where high temporal resolution is acceptable. Figures 1 to 3 illustrate the grid functions of finite volume method simulations with spatial step sizes (h) of 0.2, 0.1, and 0.05, respectively (Barth, 2018). As the spatial step size decreases, the temperature distribution becomes more accurate, capturing finer details and reducing numerical errors. However, increased precision also leads to higher computational demands.

The Crank-Nicolson method, known for its second-order accuracy in both space and time, can be used explicitly or implicitly. Figures 4 to 6 show explicit Crank-Nicolson simulations with spatial step sizes of 0.2, 0.1, and 0.05. The smallest step size (h = 0.05) provides the highest accuracy for the heat equation solution, though at a higher computational cost (Du, 2018). The implicit Crank-Nicolson method is preferred for its stability over longer time periods, as shown in Figures 10 to 12 for a maximum time of 0.05 seconds. The implicit method with a spatial step size of 0.05 is the most accurate, capturing finer fluctuations in temperature distribution.

Conclusion

The study achieved its objectives by solving the heat equation using numerical methods and compare the most suitable method between the Finite Volume Method (FVM) and the Crank-Nicolson method. Through extensive tests and simulations, it was found that the implicit Crank-Nicolson method provided more stability and better results over long intervals compared to FVM. The results showed that while the Crank-Nicolson method enhances accuracy and stability in thermal modelling, FVM produced unstable results over longer intervals. Consequently, the research concluded that the Crank-Nicolson method is superior for detailed thermal analysis and long-term simulations. To achieve more accurate and stable solutions for the heat equation, using mixed (Robin) boundary conditions is recommended for better modelling of heat interactions at boundaries compared to Dirichlet and Neumann conditions (Bollati, 2018). Additionally, employing standard finite difference methods, including forward, backward, central, and fourth-order schemes, ensures higher accuracy and stability, especially for nonuniform gradients and spatial variations. Implementing adaptive mesh refinement (AMR) locally refines the mesh in areas with high temperature gradients, enhancing solution accuracy and stability without significantly increasing computational costs (Dunning, 2020).

References:

Barth, T. a. (2018). Finite volume methods: foundation and analysis. *Encyclopedia of computational mechanics second edition*, 1-60.

Bollati, J. a. (2018). One-Phase Stefan-Like Problems with Latent Heat Depending on the Position and Velocity of the Free Boundary and with Neumann or Robin Boundary Conditions at the Fixed Face. *Mathematical Problems in Engineering, 2018*(1), 4960391.

Du, B. a. (2018). Partial differential equation modeling with Dirichlet boundary conditions on social networks. *Boundary Value Problems, 2018*(1), 1-11.

Dunning, D. a. (2020). Adaptive mesh refinement in the fast lane. *Journal of Computational Physics, 406*, 1-14.

Herbin, R. a.-C. (2023). A consistent quasi--second-order staggered scheme for the two-dimensional shallow water equations. *IMA Journal of Numerical Analysis, 43*(1), 99-143.

Islam, M. A. (2018). Numerical solution of one-dimensional heat equation by Crank Nicolson method. *Int. Conf. on Mechanical, Industrial and Energy Engineering*, 1-4.

Mojumder, M. S. (2023). Efficient Finite Difference Methods for the Numerical Analysis of One-Dimensional Heat Equation. *Journal of Applied Mathematics and Physics, 11*(10), 3099--3123.

Saptaningtyas, F. a. (2018). Finite volume method with explicit scheme technique for solving heat equation. *Journal of Physics: Conference Series, 1097*(1), 1-11.

Stabile, G. a. (2018). Finite volume POD-Galerkin stabilised reduced order methods for the parametrised incompressible Navier--Stokes equations. *Computers \& Fluids, 173*, 273-284.

Xiang, F. a. (2018). Digital twins technolgy and its data fusion in iron and steel product life cycle. *IEEE Xplore*, 1-5.

Zhu, Q. a. (2021). Heat conduction: Mathematical modeling and experimental data. *J. Emerg. Investig, 4*, 1-4.